NONLINEAR FILTRATION THROUGH A POROUS WEDGE IN THE PRESENCE OF A PHASE TRANSITION

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UDC 532.546.536.24

An approximate solution is given for the temperature distribution in a porous body bearing a flow of cooling agent which undergoes a phase transition.

We consider the passage of an incompressible liquid of constant specific heat through an isosceles porous wedge. The pressure and temperature at the faces AC and AB (BC) of the wedge (Fig. 1a) are constant and are correspondingly P_{in}; T_{in} and P_{out}; T_{out} ($P_{in} > P_{out}$; $T_{in} < T_{out}$); within the region of motion of the liquid, there is boiling, giving rise to a gas, the thermodynamic state of which defined by

$$p_2 = \rho_2 R T_2. \tag{1}$$

If the steady-state motion of the liquid within the porous medium is accompanied by a phase transition, the entire region consists of two zones separated by a transition layer; one zone AaCDA is filled solely with liquid, and the other ABCaA with vapor. The thickness of the transition layer may be taken as zero, because it is extremely small compared with the dimensions of the porous body. The temperature of the coolant changes stepwise on passage through the boiling surface; the superheating of the liquid is substantially dependent on the nature of the liquid and the roughness of the heating surface [1]. A porous medium is very rough and there is a fairly extensive class of liquids for which this temperature discontinuity is small, e.g., water, and so we neglect the superheating of the liquid on the boiling isotherm.

We adopt the following assumptions for the infiltration region:

1) the coolant temperature and the temperature of the porous medium are the same at each point [2-4];
2) the viscosities of the liquid and gas are dependent only on temperature;
3) the motion of the liquid and vapor in the porous body is subject to a power resistance law with the same power $n > 0$:

$$V_j^n \bar{V}_j = -\frac{\alpha_j}{\mu_j(T)} \operatorname{grad} p_j. \tag{2}$$

The entire wedge ABCDA receives an incompressible liquid of the same nature as that in zone AaCDA, and in exactly the same way there moves a gas of the same nature as that in zone ABCaA, i.e., we actually consider the motion of a hypothetical liquid (gas) through a porous wedge, the thermophysical characteristics of which are monotonic and continuous functions of temperature and pressure, with no phase transition. Then in the region of motion we have, by virtue of our three assumptions a temperature distribution defined by the solution to

$$\Delta T - \frac{c_j \rho_j}{\lambda_{j.\mathrm{eff}}} \bar{V}_j \operatorname{grad} T_j = 0, \tag{3}$$

that satisfies the following boundary conditions: for a liquid at face AC, $T_1 = T_{in}$; $p_1 = p_{in}$ and $T_1 = T_{1out}$; $p_1 = p_{1out}$ at faces AB and BC; for the gas at face AC, $T_2 = T_{2in}$; $p_2 = p_{2in}$ and $T_2 = T_{out}$; $p_2 = p_{out}$, at faces AB and BC. Here the constants T_{1out}, p_{1out}, T_{2in}, p_{2in} are unknowns to be determined.

Polytechnical Institute, Voronezh. Translated from Inzhenerno-Fizicheskii Zhurnal, Vol. 21, No. 5, pp. 922-928, November, 1971. Original article submitted October 26, 1970.

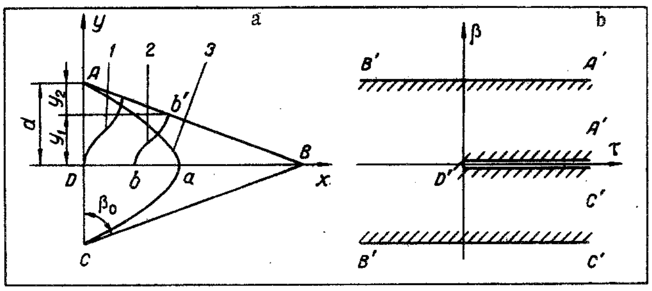

Fig.1. Filtration region in: a) x–y coordinates: 1) $\tau = 0$; τ = constant < 0; 3) phase-transition line; b) $\tau - \beta$ coordinates.

We use curvilinear orthogonal coordinates P, Ψ, and φ as shown in [5], which enables us to put (3) in the form

$$\frac{\partial}{\partial p_j}\left(\varkappa_j\frac{\partial T_j}{\partial p_j}\right)+\frac{\partial}{\partial \Psi_j}\left(\frac{1}{\varkappa_j}\cdot\frac{\partial T_j}{\partial \Psi_j}\right)+\frac{C_j}{\lambda_{j\mathrm{eff}}}\cdot\frac{\partial T_j}{\partial p_j}=0, \tag{4}$$

where $\varkappa_1 = v_1^n\mu_1(T)/\alpha_1$; $\varkappa_2 = R^{n+1}v_2^nT_2^{n+1}\mu_2(T)/\alpha_2 p_2^{n+1}$; $v_1 = V_1$; $v_2 = \rho_2 V_2$; $C_1 = c_1\rho_1$; $C_2 = c_p$; while Ψ_j is the current function that satisfies

$$v_{jx}=\frac{\partial \Psi_j}{\partial y};\; v_{jy}=-\frac{\partial \Psi_j}{\partial x}. \tag{5}$$

If the resistance law is linear (n = 0), it follows from (4) that the temperature is a single-valued function of the pressure, and so the partial differential equation (4) can be reduced to an ordinary differential equation, which is readily integrated subject to the above boundary conditions. If the infiltration is nonlinear ($n \neq 0$), it is rather more difficult to solve Eq. (4), but for the assumed range of infiltration rates we may put $v_{nj} \approx 1/\nu_j$ = const; in this case we may assume as before that

$$T_j = T_j(p) \tag{6}$$

and Eq. (4) becomes

$$\frac{1}{D_j-T_j}\cdot\frac{dT_j}{dp_j}=\frac{C_j}{\lambda_{j\,\mathrm{eff}}\tilde{\varkappa}_j}. \tag{7}$$

Here $\tilde{\varkappa}_j = \varkappa_j\nu_j/v_j^n$, D_j is a constant of integration.

We use (6) to put for the liquid and gas respectively that

$$P_1=\int\limits_{p_{\mathrm{in}}}^{p_1}\frac{dp_1}{\mu_1(T)};\; P_2=\frac{1}{R^{n+1}}\int\limits_{p_2}^{p_{\mathrm{out}}}\frac{p_2^{n+1}dp_2}{T_2^{n+1}\mu_2(T)}. \tag{8}$$

As P_j is determined apart from an arbitrary constant, we may choose the latter such that $P_j = 0$ on the faces AB and BC.

We introduce the dimensionless quantities

$$\tilde{P}_j=P_j/P_{j\,\mathrm{in}};\; \tilde{\Psi}_j=\Psi_j/v_{j0}d;\; \tilde{v}_j=v_j/v_{j0};\; \tilde{x}=x/d;\; \tilde{y}=y/d.$$

Here

$$P_{1\mathrm{in}}=\int\limits_{p_{\mathrm{in}}}^{p_{1\,\mathrm{out}}}\frac{dp_1}{\mu_1(T)};\; P_{2\mathrm{in}}=\frac{1}{R^{n+1}}\int\limits_{p_{2\,\mathrm{in}}}^{p_{\mathrm{out}}}\frac{p_2^{n+1}dp_2}{T_2^{n+1}\mu_2(T)};$$

V_{j0} is the filtration rate at point D; the constants $P_{j\,\mathrm{in}}$, v_{j0} are to be determined. From Eq. (7) we have

$$\frac{dT_j}{d\tilde{P}_j}=M_j(D_j-T_j). \tag{9}$$

($M_1 = P_{1\mathrm{in}}c_1\nu_1\alpha_1\rho_1/\lambda_{1\mathrm{eff}}$; $M_2 = P_{1\mathrm{in}}c_p\nu_2\alpha_2/\lambda_{2\mathrm{eff}}$).

We introduce new independent variables v and β related to the coordinates $\tilde{x}$ and $\tilde{y}$ by

$$d\tilde{x}_j = -\frac{\cos\beta}{\chi_j \tilde{v}_j^{n+1}}\, d\tilde{P}_j - \frac{\sin\beta}{\tilde{v}_j}\, d\tilde{\Psi}_j;$$
$$d\tilde{y}_j = -\frac{\sin\beta}{\chi_j \tilde{v}_j^{n+1}}\, d\tilde{P}_j + \frac{\cos\beta}{\tilde{v}_j}\, d\tilde{\Psi}_j, \tag{10}$$

where

$$\chi_j = v_{j0}^{n+1} d/\alpha_j P_{j\text{in}}.$$

We get the following system of equations in the coordinates $\tilde{v}$ and β after use of (2) and (5):

$$\frac{\partial\tilde{\Psi}_j}{\partial\beta} = \frac{1}{\chi_j \tilde{v}_j^{n+1}} \cdot \frac{\partial\tilde{P}_j}{\partial\tilde{v}_j};\quad \frac{\partial\tilde{\Psi}_j}{\partial\tilde{v}_j} = -\frac{n+1}{\chi_j \tilde{v}_j^{n+1}} \cdot \frac{\partial\tilde{P}_j}{\partial\beta}, \tag{11}$$

which are transformed by replacement of the variables as follows:

$$\tau_j = \frac{1}{\sqrt{n+1}} \ln\tilde{v}_j;\quad Q_j(\tau,\ \beta) = \tilde{P}_j \exp(-\varepsilon\tau_j) \tag{12}$$

and elimination of $\tilde{\Psi}_j$ to get for $Q_j(\tau, \beta)$ a Helmholtz equation

$$\Delta Q_j - \varepsilon^2 Q_j = 0. \tag{13}$$

Here $\varepsilon^2 = n/2\sqrt{n+1}$.

In the variables τ and β, the infiltration region is an infinite strip of width $2\beta_0$ with a slot along the positive semiaxis of τ (Fig. 1b). Because of this slot $Q_j(\tau, \beta)$ is a function of more than one sheet in this region, and it is therefore desirable to restrict consideration only to the upper part of the belt for $\beta_0 \geq \beta \geq 0$, within which $Q_j(\tau, \beta)$ consists of one sheet. This corresponds to consideration of infiltration within the half-wedge ABD.

At internal points in region ABD, we have finite values for $Q_j(\tau, \beta)$; although there are singularities at points A and B, $Q_j(\tau, \beta)$ is absolutely integral with respect to τ in the range from $-\infty$ to $+\infty$, as can be demonstrated from physical consideration of the singularities at points A and B. Then we can apply to $Q_j(\tau, \beta)$ a Fourier transformation with respect to τ within the infiltration region.

The following is a solution to the Helmholtz equation via an integral Fourier transform:

$$Q_j(\tau,\ \beta) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\operatorname{sh} q(\beta_0-\beta)}{\operatorname{sh} q\beta_0}\, \bar{Q}_j(\lambda,\ 0) \exp(i\lambda\tau_j)\, d\lambda, \tag{14}$$

where

$$\bar{Q}_j(\lambda,\ 0) = \int_{-\infty}^{\infty} Q_j(\tau,\ 0) \exp(-i\lambda\tau_j)\, d\tau_j. \tag{15}$$

Equation (15) contains the unknown function $Q_j(\tau, 0)$, which is deduced via the condition that the line $\beta = 0$ is a straight line.

Consider the line τ = constant within the plane of the half-wedge (Fig. 1a); the family of such lines is bounded by the curve $\tau = 0$, and the symmetry condition can be used only for $\tau \leq 0$. We find for $\tau < 0$ that y_1 and y_2 move successively from point b to point b' for τ = constant and then for β = constant move along the boundary of the wedge from b' to A. We use (10)-(12) to determine y_1 and y_2; summation of these quantities gives us an integral Fourier–Fredholm equation determining $Q_j(\tau, 0)$ for $\tau < 0$:

$$\int_{-\infty}^{\infty} \frac{q \operatorname{cth} q\beta_0}{1+q^2} \left(\frac{n+2}{2\sqrt{n+1}} + i\lambda\right) \bar{Q}_j(\lambda,\ 0) \exp i\lambda\tau_j d\lambda = \frac{2\pi\chi_j}{\sqrt{n+1}} \exp \frac{n+2}{2\sqrt{n+1}}\tau_j. \tag{16}$$

Note that $Q_j(\tau, 0) = \exp(-\varepsilon\tau_j)$ for $\tau \geq 0$ it is physically clear that $Q_j(\tau, 0)$ is monotonic also for $\tau \to -\infty$, where it tends to zero ($Q_j(0, 0) = 1$) together with a certain exponential $\exp m\tau$; we put approximately for $\tau < 0$

$$Q_j(\tau,\ 0) \approx \exp m\tau_j, \tag{17}$$

where the factor m has to be determined.

Now from (16) we obtain via the theory of residues an equation for determining the coefficient χ_j:

$$\chi_j = \left(1 + \frac{\sqrt{n+1}}{m + \dfrac{n+2}{2\sqrt{n+1}}}\right) \operatorname{ctg} \beta_0 + \frac{(n+2)\,\pi^2}{\beta_0^3} \sum_{k=0}^{\infty} \frac{(k+1)^2}{\left[\dfrac{(k+1)^2\pi^2}{\beta_0^2} - 1\right] K_n} \left(\frac{1}{m+K_n} + \frac{1}{\varepsilon + K_n}\right). \tag{18}$$

We use the theorem of residues with (14) and (17) to get the distribution of Q in coordinates τ and β:

$$Q(\tau,\ \beta) = \frac{\beta_0 - \beta}{\beta_0} \exp(-\varepsilon\tau_j) + \frac{\pi}{\beta_0^2}(m+\varepsilon) \sum_{k=0}^{\infty} \frac{(k+1)\sin\dfrac{(k+1)\pi}{\beta_0}\beta}{K_n(\varepsilon - K_n)(m+K_n)} \exp(-K_n\tau_j) \quad (\tau \geqslant 0); \tag{19}$$

$$Q(\tau,\ \beta) = \frac{\sin q_m(\beta_0 - \beta)}{\sin q_m \beta_0} \exp m\tau_j + \frac{\pi}{\beta_0^2}(m+\varepsilon) \sum_{k=0}^{\infty} \frac{(k+1)\sin\dfrac{(k+1)\pi}{\beta_0}\beta}{K_n(\varepsilon + K_n)(m - K_n)} \exp K_n\tau_j \quad (\tau \leqslant 0). \tag{20}$$

Here $K_n = \sqrt{\varepsilon^2 + \dfrac{(k+1)^2\pi^2}{\beta_0^2}}$; $q_m = \sqrt{m^2 - \varepsilon^2}$.

We use the condition $\tilde{y} = 0$ for $\beta = 0$, $\tau = -\infty$ with (10), (11), and (20) to get a formula for m:

$$\frac{q_m \operatorname{ctg} q_m\beta_0}{m - \dfrac{n+2}{2\sqrt{n+1}}} + \frac{\pi^2}{\beta_0^3}(m+\varepsilon) \sum_{k=0}^{\infty} \frac{(k+1)^2}{K_n(K_n - m)(\varepsilon + K_n)\left(K_n - \dfrac{n+2}{2\sqrt{n+1}}\right)} = 0. \tag{21}$$

The solutions to (13) for infiltration of liquid alone or of gas alone are the same, so

$$Q_1(\tau,\ \beta) = Q_2(\tau,\ \beta) \text{ and } \chi_1 = \chi_2 = \chi. \tag{22}$$

As $Q_j(\tau,\ \beta)$ is determined unambiguously, it follows from (8), (10), (12), (19), and (20) that the isobars in the infiltration region remain unchanged for any constant values of the data in this problem. Moreover, the isobars for infiltration of liquid alone coincide with the isobars for the case of gas motion provided that n is the same in both instances.

Now we can select p_{1out}, T_{1out}, p_{2in}, T_{2in} such that the solutions for the liquid in the zone $A\alpha CA$ and for the vapor in the zone $ABC\alpha A$ coincide along the line $A\alpha C$ and satisfy the conditions for the problem. The basis for this is the circumstance that in both the above cases the values of $\tilde{P}_j$ are the same, and also that the lines $\tilde{P}_j$ = const coincide approximately with the isotherms; further, on any of these lines and at each point, the gas and liquid flows and the heat flux are mutually proportional.

It is sufficient to restrict oneself to proportionality of the heat flux and coolant flow at the points of intersection of the lines $\tilde{P}_j$ = const and DB in order to solve the problem.

The following conditions should be met at the point where the boiling isotherm meets the DB axis:

$$1)\ T^* = F(p^*);\ 2)\ T_1^* = T_2^*;\ 3)\ p_1^* = p_2^*;\ 4)\ v_{2x} = \rho_1 v_{1x};\ 5)\ \lambda_{2\text{eff}} \frac{\partial T_2}{\partial x} - \lambda_{1\text{eff}} \frac{\partial T_1}{\partial x} = L\rho_1 v_{1x};\ 6)\ x_1^* = x_2^*. \tag{23}$$

Consider condition (6) in Eq. (23); we get from Eq. (10) for the Dx axis that

$$d\tilde{x}_j = -\frac{1}{\chi_j} \exp\left(-\frac{n+2}{2\sqrt{n+1}}\tau_j\right)\left[dQ_j(\tau,\ 0) - \varepsilon Q_j(\tau,\ 0)\,d\tau_j\right].$$

This means that to meet condition (6) in Eq. (23) we have

$$\tau_1^* = \tau_2^* = \tau^*. \tag{24}$$

Then condition (4) in Eq. (23) gives

$$v_{20} = \rho_1 v_{10}. \tag{25}$$

From (22) we have

$$\alpha_1 P_{1\mathrm{in}} \rho_1^{n+1} = \alpha_2 P_{2\mathrm{in}}. \tag{26}$$

Condition (5) of (23) is then put as follows:

$$c_1 (D_1 - T^*) \rho_1^{-(n+1)} - c_p (D_2 - T^*) = L. \tag{27}$$

Condition (1) of (23) is the equation for the P – T phase diagram for the liquid.

From Eqs. (7) and (9) we get expressions for $T_{2\mathrm{in}}$, $T_{1\mathrm{out}}$, $p_{2\mathrm{in}}$, $p_{1\mathrm{out}}$:

$$T_{2\mathrm{in}} = D_2 - (D_2 - T^*) \left(\frac{D_1 - T_{\mathrm{in}}}{D_1 - T^*} \right)^{\nu_2 c_p \rho_1^n / \nu_1 c_1}; \tag{28}$$

$$T_{1\mathrm{out}} = D_1 - (D_1 - T_{\mathrm{in}}) \left(\frac{D_2 - T_{\mathrm{out}}}{D_2 - T_{2\mathrm{in}}} \right)^{\nu_1 c_1 / \nu_2 c_p \rho_1^n}; \tag{29}$$

$$p_{1\mathrm{out}} = p_{\mathrm{in}} + \frac{P_{1\mathrm{in}}}{M_1} \int\limits_{T_{\mathrm{in}}}^{T_{1\mathrm{out}}} \frac{\mu_1(T)}{D_1 - T} \, dT; \tag{30}$$

$$p_{2\mathrm{in}}^{n+2} = p_{\mathrm{out}}^{n+2} - \frac{(n+2) P_{2\mathrm{in}}}{M_2} \int\limits_{T_{2\mathrm{in}}}^{T_{\mathrm{out}}} \frac{T_2^{n+1} \mu_2(T)}{D_2 - T} \, dT. \tag{31}$$

The constants D_j are deduced from the boundary conditions for each zone:

$$\frac{M_1}{P_{1\mathrm{in}}} (p^* - p_{\mathrm{in}}) = \int\limits_{T_{\mathrm{in}}}^{T^*} \frac{\mu_1(T)}{D_1 - T} \, dT; \tag{32}$$

$$\frac{M_2}{(n+2) P_{2\mathrm{in}}} \left[p_{\mathrm{out}}^{n+2} - (p^*)^{n+2} \right] = \int\limits_{T^*}^{T_{\mathrm{out}}} \frac{T^{n+1} \mu_2(T)}{D_2 - T} \, dT. \tag{33}$$

Then (19) and (20), together with (8) and (25)-(33), define the temperature and pressure distributions in $\tau - \beta$ coordinates when an incompressible liquid infiltrates a porous wedge and boils there. The conversion to x – y coordinates is performed numerically using Eq. (10).

NOTATION

V	is the velocity of filtration;
V_x, V_y	are its projections on the coordinate axes;
β	is the angle of inclination of the filtration velocity vector to the axis Dx;
p	is the pressure;
T	is the temperature;
α	is the power-law filtration coefficient;
ρ	is the density;
R	is the gas constant;
c_1	is the heat capacity of the liquid;
c_p	is the capacity of the vapor at constant pressure;
λ_{eff}	is the mean thermal conductivity of the liquid and the porous medium;
L	is the latent heat of vaporization;
λ	is the Fourier parameter;
d	is the characteristic dimension;

$q = \sqrt{\lambda^2 + \varepsilon^2}$.

Subscripts

*	denotes parameters on the boiling isotherm;
$j = 1, 2$;	
1	denotes parameter characterizing the liquid;
2	denotes parameters of the gas (vapor).

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